Chapter 4
NOISE AND RANDOM SIGNALS

One of the foremost-used Electronic Attack jamming methods is to emit radio noise in the frequency band of a target receiver. In an un-jammed environment, noise is the fundamental limiting factor which prevents accurate reception of a radio signal. Jammers often mimic noise signals because of the devastating impact of noise upon radio receivers. The purpose of this chapter is to gain an understanding of the characteristics of noise and its effect upon reliable radio reception. How jammers create noise signals will be discussed in Chapter xxx.

We think of noise as a sound which is irritating to our sense of hearing. *Electrical noise* had an original meaning of an electrical disturbance to a radio signal which resulted in noise when we listened to the signal on a radio. Electrical noise is due to a corruption of the original signal. With ensuing technological advances, the term noise has come to represent all *unintentional* electrical signals which undermine the reliable reception of transmitted signals. In addition to the noise that we hear on audio signals, noise is also evident as snow and specks on video TV signals, and causes reception errors of digital signals such as digital voice modulation and data links.

As an introduction to the effects of radio noise, take the transmitted signal to be a sinusoid as shown in the top plot of Figure 4-1. The middle plot shows a noise signal which will combine with the sinusoid. The received signal in the presence of noise might appear as the sum of the two signals as shown in the bottom plot of the figure. Notice that the noise signal of the middle plot is random and unpredictable. Since the noise signal is random, we cannot predict its waveform. Even if we know

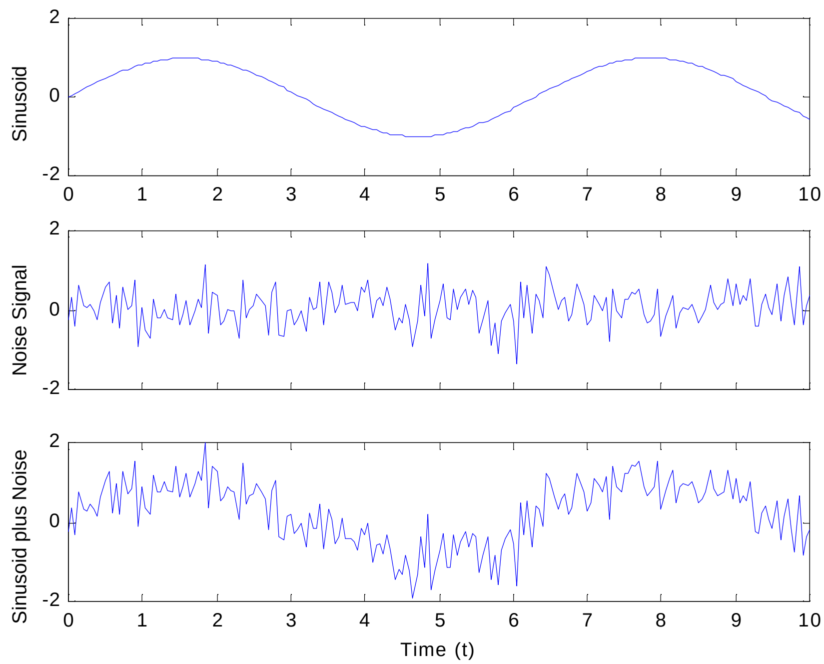


Figure 4-1. Effects of noise on signal. Top plot shows transmitted sine wave, middle plot is additive noise, and bottom plot is the result when the additive noise is combined with the sinusoid.

in advance what the frequency content of the noise signal is, the signal of interest might be in the same band, preventing its removal by filtering.

Random signals cannot be described analytically, i.e., we cannot write a mathematical expression for the noise voltage or current. Instead, we evaluate random signals statistically, and characterize them by their mean or average value, standard deviation, bandwidth, etc. This will be explored in Section 4.3.

Those signals which *unintentionally* corrupt the electrical signal but are *intentionally* generated, are called interference, rather than noise (e.g., two radio signals at the same frequency). Jamming signals differ in that they are intentionally generated with the intent to corrupt target signals.

Noise is of concern at the receiver rather than at the transmitter. It is only when the signal strength is relatively weak that the noise has enough power to impact upon the signal. It is at the receiver that the signal strength can have been reduced to a vulnerable level.

The noise which affects the received electrical signal can be generated in *nature* or it can be *man-made*. Examples of each are galactic noise, coming from the sun and stars, and automobile ignition noise. We also classify noise as either *external* or *internal* to the receiver. Those noise sources which are external to the equipment cannot be eliminated or diminished. The radio engineer and operator can at times diminish the effects of external noise if they are aware of the frequency ranges where the sources of external noise are present and if they can operate at other frequencies. We will now discuss some common noise sources in order to gain an understanding of noise characteristics.

4.1 EXTERNAL NOISE SOURCES

4.1.1 Atmospheric Noise

Lightning from thunderstorms contributes considerable amounts of noise at frequencies from a few Hz up to about 20 MHz, with the noise amplitude decreasing with increasing frequency. HF radio signals can be severely degraded by atmospheric noise, but it is usually not significant at VHF and higher frequencies.

Depending upon the severity of the storm and the frequency of occurrence, a statistical mean can be generated to indicate the expected value of the noise signal. There are world-wide maps of statistical modeling of this recurrence and severity of lightning storms.

4.1.2 Cosmic Noise

Cosmic noise has origins external to the terrestrial atmosphere. Primary sources include solar noise from the sun, galactic noise from the Milky Way, and other discrete cosmic sources such as the intense star Cassiopeia A.

Noise from cosmic sources must pass through the earth's ionosphere and atmosphere before being collected by radio reception antennas on the earth's surface. The absorptive properties of the ionosphere prevent this noise from penetrating and reaching the earth at frequencies below about 20 MHz. Molecular absorption of the atmosphere likewise prevents frequencies above about 10 GHz from reaching earthbound antennas. Space-borne antennas, i.e., those mounted on satellites, above about 1000 km do not benefit from these atmospheric absorption mechanisms so receive noise over a much broader band of frequencies. As with atmospheric noise, the magnitude of cosmic noise decreases with frequency.

It should be noted that, in general, cosmic noise is not a problem for the receiver if the antenna can be pointed away from the source of the noise. However, this is not always possible since the transmitted signal and the noise source may originate in the same direction from the receiver.

4.1.3 Man-Made Noise

Noise from man-made sources also has the potential to degrade a received signal. The frequencies over which these sources contribute noise varies from a few Hz up to about a GHz. The problems are more severe in urban and suburban areas because of the relatively increased density of noise generators in these areas. Examples of man-made noise sources are ignition systems from gasoline engines, corona noise from high-voltage power lines, gap noise from utility distribution lines, welders, plastics industries (who use microwave energy to heat the plastics), and many other devices and equipment in industry and within homes.

4.2 INTERNAL NOISE SOURCES

In electrical equipment noise is generated from several sources within the circuits themselves. Probably the most important of these, the one that communications engineers spend the most time trying to control, is thermal noise, which is produced by random motions of electrons. Other types of noise in circuits are shot noise, which is caused by random fluctuations in current flow, and flicker noise from transistors. Thermal noise is the standard against which all other noise types are compared, so we will examine it in detail.

4.2.1 Thermal Noise

When the atoms of any material receive energy in the form of heat, their electrons become agitated and attempt to escape their bonds with their nuclei. In a conducting material, there are many free electrons which can move in any direction within the material. The direction which an individual electron moves is random. Movement of electrons constitutes a current flow, so that if a complete circuit is formed current will flow randomly through the circuit. The average current flow will be zero, since current is just as probable to flow in one direction as another.

With a random current flow caused by applied thermal energy to a resistor, a random voltage will be developed across the resistor. It has been determined (see for example Rosie, 1966) that the RMS value (see Section 2.6) of this noise voltage across a resistor is given by

$$V_n = \sqrt{4RkTB_n}, \tag{4-1}$$

where the n subscript indicates noise, R is the resistance of the resistor, k is Boltzmann's constant

$$k = 1.38 \ x \ 10^{-23} \ \frac{joules}{^\circ kelvin}, \tag{4-2}$$

T is the temperature in degrees kelvin, and B_n is the noise bandwidth. The kelvin temperature scale is related to the centigrade scale in that a one degree change kelvin, K, is equivalent to one degree change centigrade, C, and

$$^\circ K = {^\circ C} + 273. \tag{4-3}$$

The kelvin scale is the *absolute* temperature scale–0 K is absolute zero, the absence of all heat.

We can model the resistor, and its thermally controlled random voltage, as shown in Figure 4-2.

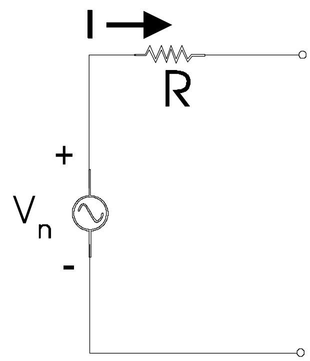


Figure 4-2. Resistor noise model.

We know the RMS value of the noise voltage, V_n, given by Equation 4-1, and we also know that its DC value is zero. But what is the power available at the output terminals of the modeled resistor shown in Figure 4-2? To see how much power can be delivered by the thermal noise of the resistor, let's connect a load resistor to the output terminals as shown in Figure 4-3. The maximum power transfer occurs when the load resistance is equal to the source resistance. If we define power available to be equal to maximum power transfer, we can find the power available out of the resistor by setting the load resistance equal to that of the resistor and solving for the power.

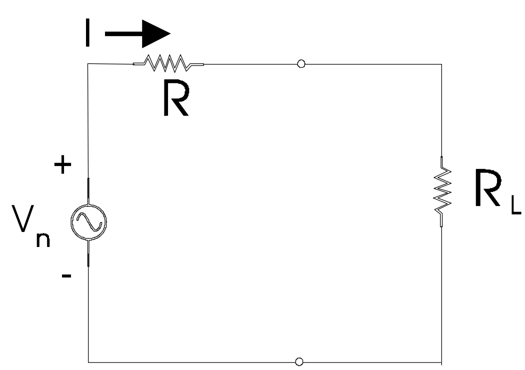


Figure 4-3. Resistor noise model connected to load resistor.

With $R_L = R$, the current in the circuit is

$$I_n = \frac{V_n}{R + R} = \frac{\sqrt{4RkTB_n}}{2R}. \tag{4-4}$$

The power in the load from the noise is therefore

$$P_n = I_n^2 R = \left(\frac{\sqrt{4RkTB_n}}{2R} \right)^2 R = kTB_n. \tag{4-5}$$

From this derivation we see that the power available is a function of the bandwidth of the system to which the noise voltage is applied. Implicit in this statement is that the output frequencies from the thermal noise extend with equal amplitude across all frequencies, i.e., the bandwidth of the noise source is infinite. While is not true in an absolute sense, it is true in a practical sense as we will see in the next paragraph. We refer to noise which has a constant amplitude at all frequencies as *white noise*. This is in allusion to light, because white light contains all colors (i.e., all frequencies of the color spectrum).

To see that thermal noise is white in a practical sense, we examine how thermal power is distributed in frequency, i.e., the power spectral density (PSD) of thermal noise. From the work by Nyquist (1928) it has been determined that the thermal noise PSD is given by

$$S_{N_T}(f) = \left(\frac{hf}{e^{hf/kT} - 1} \right) + hf \quad \frac{watts}{hertz}, \tag{4-6}$$

where h is Planck's constant,

$$h = 6.63 \times 10^{-34} \quad joule \cdot seconds. \tag{4-7}$$

We see from Equation 4-6 that the noise power is indeed distributed in frequency, but it is difficult to visualize this distribution from the equation. If we set the temperature to 290 K (room temperature), and plot S_N versus f we obtain the result shown in Figure 4-4 below.

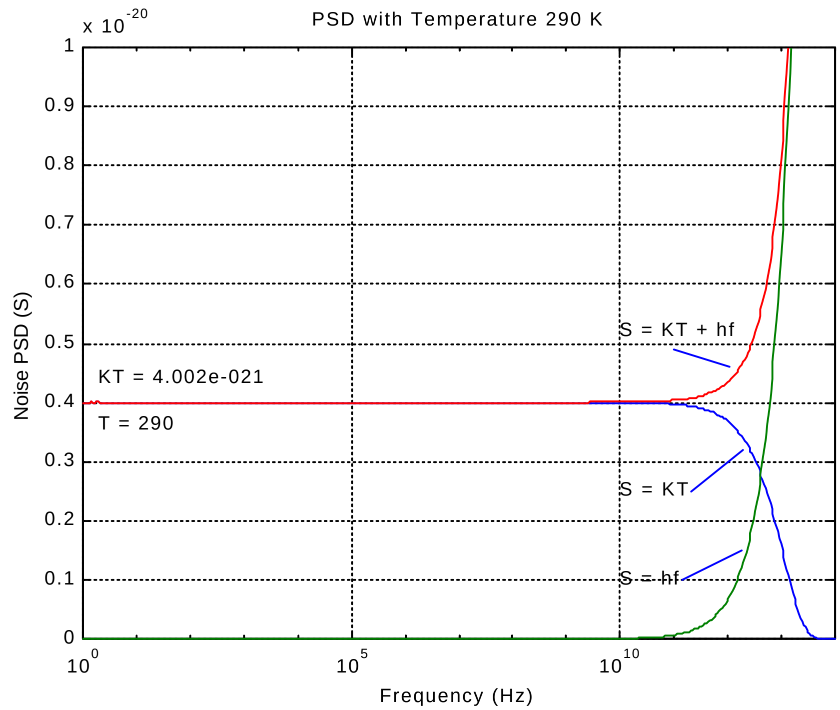


Figure 4-4. Plots of the components of Equation 4-6. Shown is that Noise PSD is constant with value kT until approximately 10^{12} Hz.

There are three separate plots shown in the graph: those contributed by the two separate terms of Equation 4-6 and their sum. The curve which represents the contribution from the second term of the equation, indicated as hf, does not influence the PSD value until about 10^{12} Hz. At about this same frequency, the contribution from the first term diminishes, as shown in the plot labeled kT. The total PSD, shown in the line labeled kT + hf, is therefore approximately equal to the first term for frequencies less than 10^{12} and to the second term for frequencies greater.

With a limit of $f = 10^{12} = 1000$ GHz (well above RF), we conclude that we are constrained to system operation where we must consider only the contributions from the left term of Equation 4-6, where we see that the PSD frequency response is flat, or white. For RF we can then state the PSD as

$$S_{N_T}(f) = \left(\frac{hf}{e^{hf/kT} - 1} \right) \quad \frac{watts}{hertz}. \tag{4-8}$$

To see if this equation can be simplified, we note from Equation xxx that e^x can be represented by the series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots \tag{4-9}$$

If x is small then $e^x \approx 1 + x$. If we set $x = hf/kT$, then for a constant temperature its only variable is f. In this case, x will be small if $f \ll kT/h$. At $T = 290$, for x to be small we require that $f \ll 6 \times 10^{12}$, so for RF, we can approximate the exponential of Equation 4-8 as $1 + hf/kT$. Making this substitution we have

$$S_{N_T}(f) \approx \left(\frac{hf}{(1 + hf/kT) - 1} \right) = \frac{hf}{hf/kT} = kT \quad W/Hz. \tag{4-10}$$

We notice in Figure 4-4 above, that the flat portion of the PSD is indeed equal to kT. For a given bandwidth, the power delivered will be the product of the PSD and B, confirming Equation 4-5 at radio frequencies, i.e., the thermal power delivered by resistive sources is given by kT multiplied by the bandwidth of the device measuring the power (i.e., absorbing the power), or

$$P_N = kTB \quad W. \tag{4-11}$$

At any given temperature T, kT is a constant. This product has units of watts per hertz and is given its own symbol, N_0, i.e.,

$$N_0 = kT. \tag{4-12}$$

As an example of computing thermal noise power, suppose that we wish to find the power available to drive a power meter connected across a 100-ohm resistor. Suppose that the power meter has a noise bandwidth of 1 kHz and the environment is at room temperature. Room temperature is usually taken as 290 degrees K. From Equation 4-11 the power available is

$$P_n = kTB_n = (1.38 \ x \ x10^{-23})(290)(10^3) = 4 \ x \ 10^{-18} \ W. \tag{4-13}$$

This is not a lot of power, so one might wonder why we are concerned with this minuscule amount. At the input of a receiver, the received signal power may have this same order of magnitude. In order to process the incoming signal, the magnitude of the signal power must be on the order of at least ten times the magnitude of the noise power. We will explore this further when we look at receivers in Chapter xxx.

4.2.2 Shot Noise

Another type of noise which is considered to be white is that of *shot noise*. The frequencies observed in this noise varies from a few Hz into the GHz region. The generation of shot noise is through a different physical mechanism than that of thermal noise, however.

Shot noise is due to the random flow of electrons through devices such as tubes, transistors, and diodes. Since the current flow can be divided into discrete charge-carrying events, i.e., through electron or hole flow, we can visualize that the current crossing a semiconductor device junction is not constant. The number of holes or electrons crossing a junction at any given moment of time is random. This can be thought of as a series of pulses of charge crossing the boundary as a function of time.

The name “shot noise” came from the vacuum tube days when the shot noise was caused by the random bombardment of the anode by electrons, as if they were shots fired from a shotgun.

4.2.3 Flicker Noise

At low frequencies the internal receiver noise power is greater than that expected from the combination of thermal and shot noise. The cause of this additional noise is controversial, but it is universally called flicker noise. It is thought to originate due to imperfections in the lattice structure of semiconductors such as diodes and transistors.

This low frequency flicker noise is not white. Its power level falls off with increasing frequency. It is observed that flicker noise PSD is inversely proportional to frequency, i.e., $S_N \sim 1/f$. For this reason flicker noise is often called 1/f noise.

4.3 RANDOM SIGNALS AND PROBABILITY

The characteristics pertaining to the thermally generated white noise signals described in Section 4.2 consist of the average value, RMS value, and PSD, but nothing else. While these values are useful, other details such as ranges of instantaneous values are not known. Additionally, other types of noise, such as those described in Section 4.1, are not necessarily white with a constant-valued PSD, nor have we developed a systematic method to determine these important parameters for an arbitrary random noise signal.

Since random signals cannot be described by analytic expressions, we must use other methods to characterize them. The branch of mathematics ready-made to describe random signals is that of probability theory. Using statistical methods we will be able to ascertain signal characteristics such as the average, standard deviation, probability of a given voltage level, and PSD. These elements of the noise signal will be useful is determining the magnitude of the deleterious effects of a particular noise signal against an individual radio receiver.

4.3.1 Elementary Probability Theory

The field of probability theory is vast and complex. A description of this entire field is beyond the purpose and scope of this exposition. Instead, a few important concepts which meet our requirements will be illustrated and quantified. The first of these concepts is the probability that a particular event will occur.

4.3.1.1 Event Probability

Most of us have a rudimentary understanding of the concept of probability. For example, it is commonly understood that if a coin is tossed, it is equally probable that a heads or a tails will appear. In quantified terms, the probability is 50% or 0.5 for each. For purposes of discussing probability, the tossing of a coin is called an experiment and the result is an event. Another experiment is the drawing of a card at random from a deck. Since each card has the same probability of being drawn, the probability of the event of drawing, say, the 3 of clubs is 1/52, the same as any other card.

The most intuitive description of event probability is that of relative frequency of occurrence. Suppose several separate events can occur, but not simultaneously, as the result of an experiment. An example would be that one and only one of the six sides of each die can appear when a pair of dice is thrown. The number that appears is the event and the tossing of dice is the experiment. We can call the list of possible events A, B, C, ... For a die an A would correspond to a one, B to a two, and so on. Suppose further that N experiments are performed and the ensuing events are noted and recorded. The number of times events A, B, ... occur can be counted and recorded as N_A, N_B, ... which can be used to determine the frequency of occurrence.

If N is large (for theoretical purposes we let $N \to \infty$), the probability of a particular event is given by the number of times the particular event occurs divided by the number of experiments, i.e., for the probability of, say, event C occurring can be found as N_C/N. This is the relative frequency of event C. Calling the probability of a particular event P, we find that

$$P\{event\} = \lim_{N \to \infty} \frac{N_{event}}{N}. \qquad (4\text{-}14)$$

It is seen that probability is always a number between zero and one. For the die, the probability of getting a 3 is the same as getting any of the other numbers 1, 2, 4, 5, or 6, which is 1/6. For shorthand, the probability notation can be stated as P{1} = P{2} = P{3} = P{4} = P{5} = P{6} = 1/6. However the probability of getting a 7 is zero, the impossible event. The probability of getting a 1 or a 5 is P{1 + 5} = 1/3 where "+" signifies "or". The probability of getting a 1, 2, 3, 4, 5 or a 6 is P{1 + 2 + 3 +4 + 5 + 6} = 1, the certain event.

For the numbers on the die, each is equally likely or probable to appear. Another example will show that equal event probability is not always the case. Suppose a bag has 10 coins: 2 half-dollars, 5 quarters, 2 dimes, and a nickel. If we were to set up an experiment where one coin was picked at random from the bag, what is the probability of drawing a coin of a given value? Using Equation 4-14 the probabilities for the different coins are P{50¢} = 0.2, P{25¢} = 0.5, P{10¢} = 0.2, and P{5¢} = 0.1.

Probability is a measure of how likely an event is to occur. Its use can be quite useful in determining the likelihood of a particular random event. Often it is desired to determine the probability of a range of events occurring. For this it is more useful to develop probability density.

4.3.1.2 Probability Density Function

In the above examples it was seen that the probabilities were distributed over particular event values. In the case of the dice, the probabilities were distributed uniformly over the events while with the coins the distribution was not uniform but favored the drawing of a quarter. This distribution of probability is called probability density.

To illustrate the usefulness of the concept of probability density suppose 1000 persons attending a ball game are picked at random and their heights measured to the nearest 0.05 of a foot (6/10 of an inch). Since the sample chosen to measure is comprised by males, females, adults, and children, a wide range of measurements is obtained. The results of this experiment are plotted in Figure 4-5. It is obvious from the plot that the measured heights are random. It is difficult (if not impossible) to

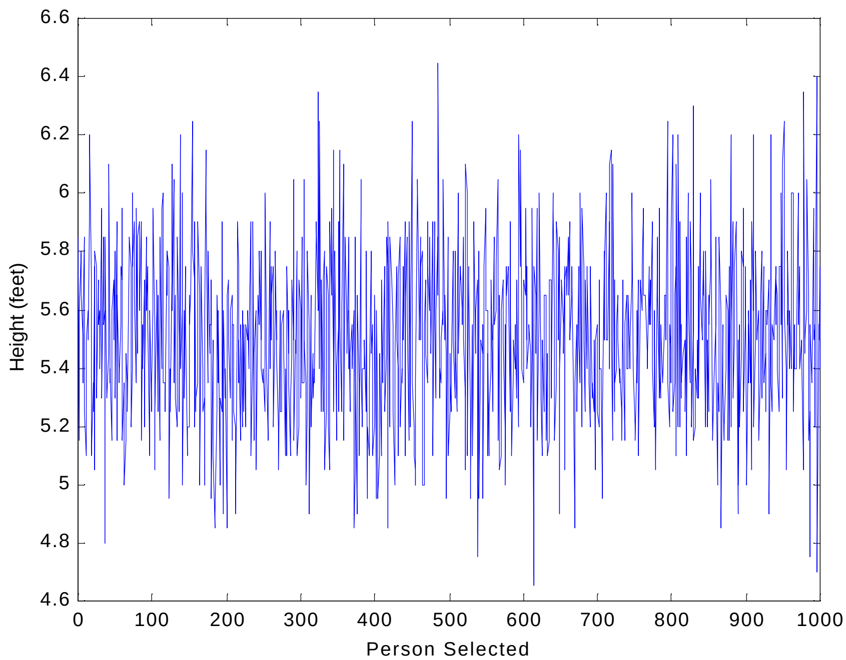


Figure 4-5. Plot of heights of 1000 people selected at random.

determine the number of persons measured at a particular height from this plot. A more helpful method is to plot the data arranged by height instead of in the order of selection. A plot of the height data with the x-axis increasing by height of those measured is shown in Figure 4-6. We could more easily find the number of persons of a particular height using this plot. As an example say it is desired to know how many persons measured 6 feet even. From the plot we see that eighteen people measured 6 feet.

We can now find the probability that a person selected at random will measure exactly six feet. Using Equation 4-14, the probability is given by the relative frequency of measuring six-foot individuals, i.e.,

$$P\{6\ ft\} = \frac{N_{6\ ft}}{N} = \frac{18}{1000} = 0.018. \tag{4-15}$$

Therefore, the probability of someone selected at random from this group of 1000 persons measuring six feet even, is 1.8%.

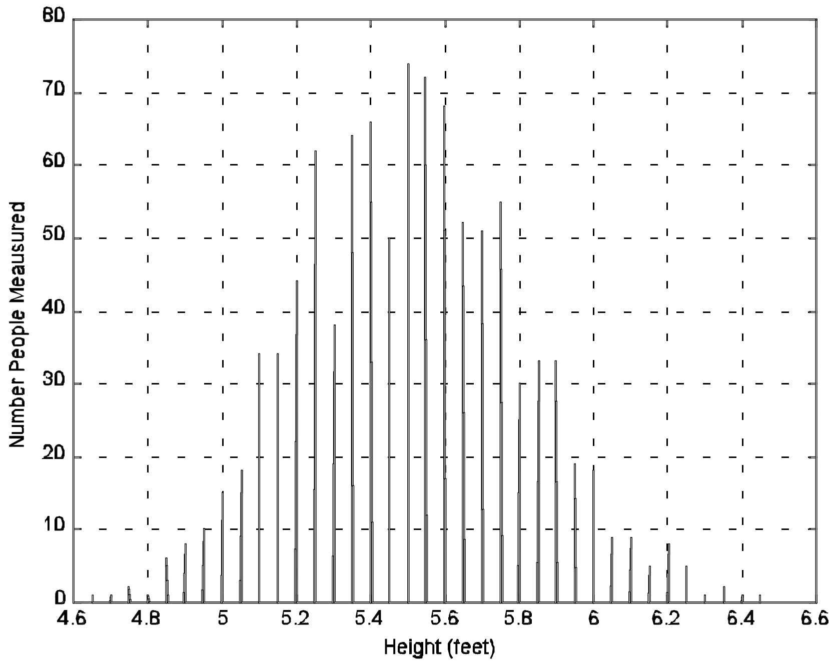


Figure 4-6. Number of persons measuring a given height, plotted by height.

Now suppose that the probability of a range of heights is desired, say from 6.0 to 6.2 feet. We can use the same plot and simply sum the contributions from those heights, giving

$$P\{6.0 - 6.2 \ ft\} = \frac{N_{6 \ ft}}{N} + \frac{N_{6.05 \ ft}}{N} + \frac{N_{6.1 \ ft}}{N} + \frac{N_{6.15 \ ft}}{N} + \frac{N_{6.2 \ ft}}{N} = \sum_{n=6.0}^{6.2} \frac{N_n}{N}$$

$$= \frac{1}{N} \sum_{n=6.0}^{6.2} N_n = \frac{1}{1000} (18 + 9 + 9 + 5 + 8) = \frac{49}{1000} = 0.049. \tag{4-16}$$

It is apparent from Equations 4-14 – 4-16 and Figure 4-6 that the number of occurrences plot can be transformed to a relative frequency/probability plot just by dividing the number of occurrences by N. Dividing the numbers of Figure 4-6 by 1000 yields the height probability plot of Figure 4-7.

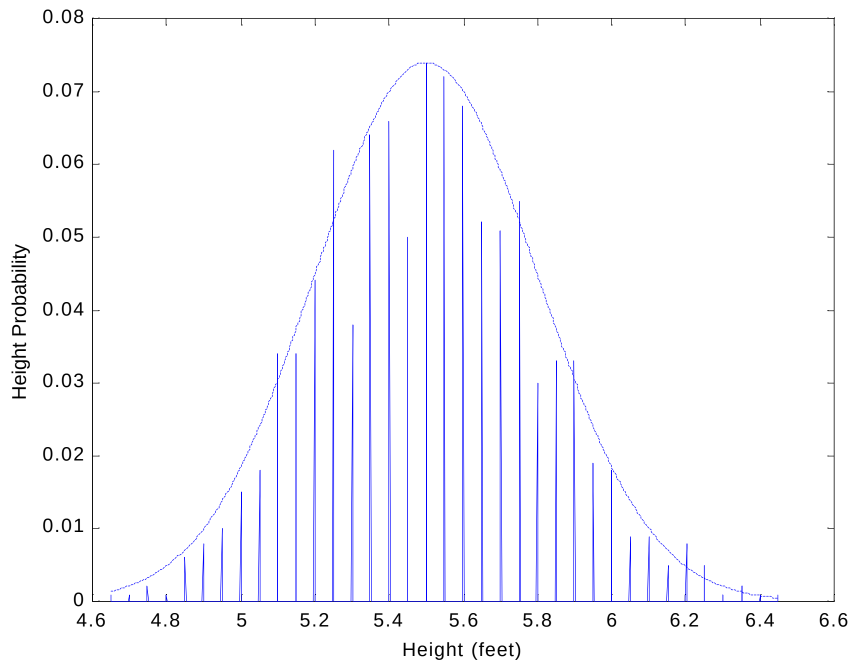


Figure 4-7. Probability of selecting a person of a given height from the group of 1000 persons. A dashed gaussian curve is superimposed over the probability lines.

Notice the smooth dashed line which traces out an approximation of the envelope of the height probabilities in Figure 4-7. This line could represent what would be expected of the height probability distribution if an infinite number of persons were measured and the data were not rounded off. With $N \rightarrow \infty$ and no rounding, any and all height values could be measured between the shortest and tallest individuals; the probability curve would become continuous with an infinite number of points between the shortest and tallest heights. With an infinite number of points, the application of Equation 4-14 results in a probability of zero for any given height.

Although it appears that the dashed line was drawn to fit the data, actually it is the other way around. The curved line represents a probability distribution known as “normal” or “gaussian” *probability density function (pdf)*. The reason that the height probabilities match the gaussian curve so well is that the height data is distributed normally or gaussian.

The gaussian pdf is the most prevalent, and therefore the most important, probability distribution found in nature. For example, if we plot the thermal noise data from the middle plot of Figure 4-1 as a relative frequency, as shown in Figure 4-8 with the gaussian pdf overlaid, we see that thermally-generated noise has a probability density function that is also normal or gaussian.

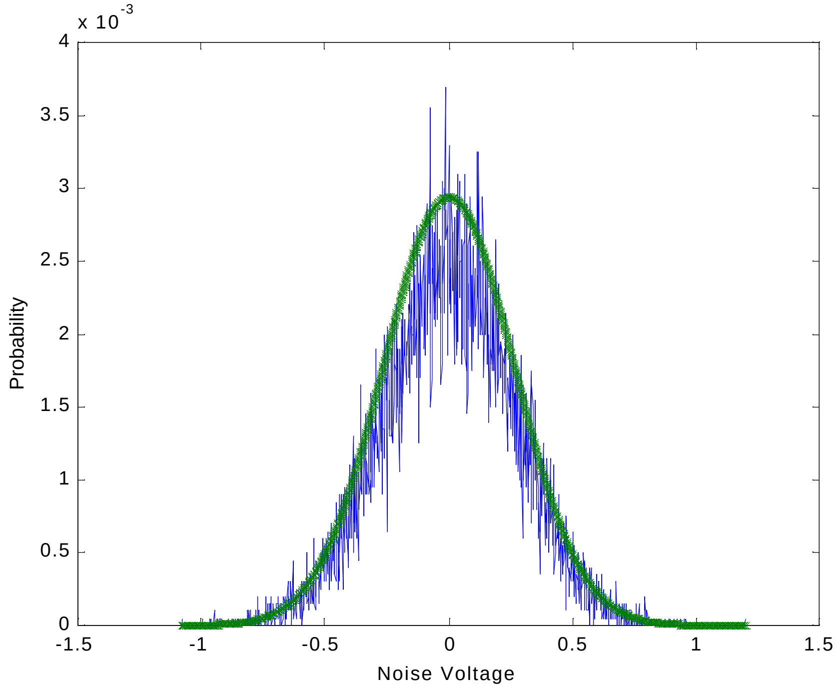


Figure 4-8. Thermal noise relative frequency plot with gaussian pdf overlaid.

It is seen that if it is known in advance how the noise voltage is distributed, the pdf can be used to predict the noise voltage probability. If the pdf is a continuous function, integration must be used vice summation as in Equation 4-16 to compute probability over a range. In Section 4.4 we will describe mathematically the gaussian and several other probability density functions. But before we can understand those descriptions we must first discuss averages and signal spread.

4.3.2 Statistical Mean, Expected Value, Variance, and Standard Deviation

Since random signals cannot be described analytically we use statistical measures to characterize them. A notation we will adopt is to use a capital letter to represent a random signal, e.g., X, while any particular value found from that random signal will be represented by its lower case, i.e., x. This will become clear as we use this notation in the following passages.

4.3.2.1 Statistical Mean

The first of these measures we will discuss is that of the statistical mean or average. Most of us have an intuitive feel for the concept of averages, which should facilitate this current discussion that we begin with an example.

In Section 4.3.1.1 we discussed a bag which contained 2 half-dollars, 5 quarters, 2 dimes, and a nickel. The average value of the coins can be found by summing the coin values and dividing by the number of coins, i.e.,

$$\begin{aligned} \textit{average}\ ¢ &= (50 + 50 + 25 + 25 + 25 + 25 + 25 + 10 + 10 + 5)/10 \\ &= 25. \end{aligned} \tag{4-17}$$

Recall that the probabilities for the different coins are P{50¢} = 0.2, P{25¢} = 0.5, P{10¢} = 0.2, and P{5¢} = 0.1. We can use these probability values to develop a easier method to find the average. Since the probabilities represent the relative frequency of occurrence, the dividing by the number of coins and the summing of multiple coins of the same value can be eliminated. Instead, we can sum each coin value multiplied by its probability to find the average,

$$\begin{aligned} \textit{average} &= 50¢ \cdot P\{50¢\} + 25¢ \cdot P\{25¢\} + 10¢ \cdot P\{10¢\} + 5¢ \cdot P\{5¢\} \\ &= 50¢ \cdot 0.2 + 25¢ \cdot 0.5 + 10¢ \cdot 0.2 + 5¢ \cdot 0.1 \\ &= 10¢ + 12.5¢ + 2¢ + 0.5¢ = 25¢, \end{aligned} \tag{4-18}$$

just as before.

As mentioned we use the terms average and mean interchangeably; we give this operation the symbol μ. If we are interested in finding the mean of random signal X, then μ = mean(X). In this example the collection of coins is X while an individual coin value is x. With this notation, we can generalize the results of Equation 4-18 as

$$\mu = mean(X) = \sum_{n=1}^{N} x_n \, P\{x_n\}, \tag{4-19}$$

where N is the number of possible outcomes and x_n is a particular outcome. (In this example N = 4 and the outcomes are x = 5¢, 10¢, 25¢ or 50¢.)

In Section 4.3.1.2 we introduced the probability density function which we saw was a continuous function. Since it is continuous, the probability of any given outcome is zero so that Equation 4-19 will not give satisfactory results for the pdf. But, since the function is continuous we recognize that the summation becomes an integration. Calling the probability of any given x-value f(x) (i.e., f(x) = P{x}), for a continuous random signal, Equation 4-19 becomes

$$\mu = mean(X) = \int_{-\infty}^{\infty} x \; f(x) \; dx. \tag{4-20}$$

In the next section we will see that the mean or average value is a special case of the expected value.

4.3.2.2 Expected Value

One way of conceptualizing the expected value is to apply the literal meaning. For example, if one coin is drawn at random from the bag described above, what coin value is expected? It should be clear that one would expect to draw a quarter from the bag since its drawing has the highest probability. However, this definition of the expected value is misleading because the expected value does not always coincide with the value of highest probability. In fact, the expected value need not be contained in X.

A better way to envision the expected value is that it represents the “center of mass” of the x values. This alludes to this same concept in classical physics which we will review for clarity. Take for discussion two masses at opposite ends of a children’s teeter-totter. Assume the masses are equal, i.e., they have the same weight. The center of mass of this system is easily seen to be at the center of the system–at the fulcrum. This is shown in graphical form in Figure 4-9 where we have

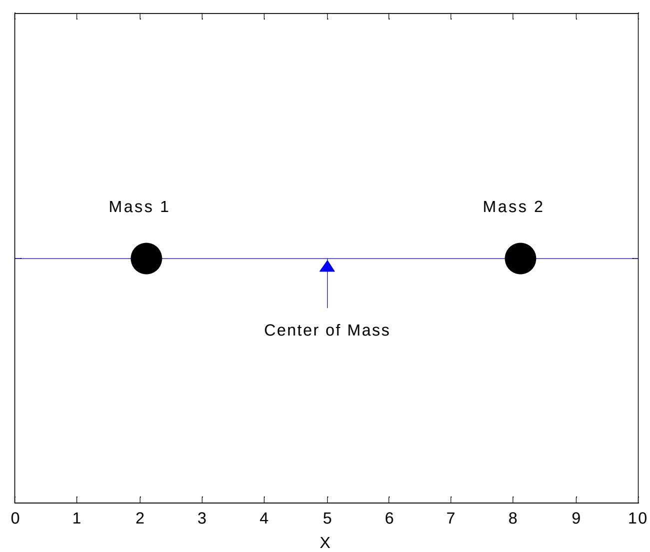


Figure 4-9. Center of mass centered between two equal masses.

arbitrarily placed the masses at X positions of 2 and 8. The center of mass can be seen to be the point where the system is in balance (at position X = 5). The point of balance is the position where the torque (= weight times distance) to the left is equal to the torque to the right.

Formalizing the center of mass mathematically, we find

$$center\ of\ mass = \frac{x_1 \cdot m_1 + x_2 \cdot m_2}{m_1 + m_2}, \tag{4-21}$$

where m_1 and m_2 are the two masses and x_1 and x_2 are their positions. For the example shown in Figure 4-9 $m_1 = m_2 = m$, $x_1 = 2$, and $x_2 = 8$, giving

$$center\ of\ mass = \frac{2 \cdot m + 8 \cdot m}{m + m} = 5, \tag{4-22}$$

which agrees with the graphical solution.

Now suppose we have three masses, as shown in Figure 4-10, with values $m_1 = 6$ located at position 1 ($x_1 = 1$), $m_2 = 2$ with $x_2 = 3$, and $m_3 = 1$ with $x_3 = 6$.

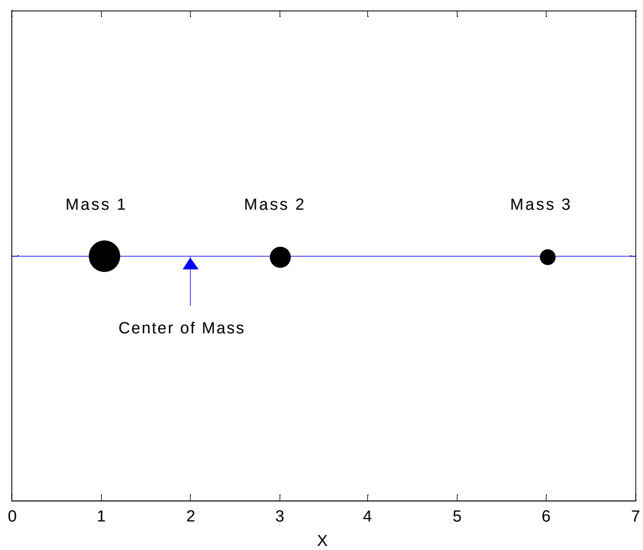


Figure 4-10. Center of mas for three unequal masses.

Extending Equation 4-22 for three masses, the center of mass is found as

$$\begin{aligned} center\ of\ mass &= \frac{x_1 \cdot m_1 + x_2 \cdot m_2 + x_3 \cdot m_3}{m_1 + m_2 + m_3} \\ &= \frac{1(6) + 3(2) + 6(1)}{9} = 2. \end{aligned} \tag{4-23}$$

Generalizing the foregoing results, the center of mass for an arbitrary number of masses can be seen to be

$$center\ of\ mass = \frac{\sum_{n=1}^{N} x_n \cdot m_n}{\sum_{n=1}^{N} m_n} = \frac{\sum_{n=1}^{N} x_n \cdot m_n}{M}, \tag{4-24}$$

where N is the number of and M is the sum of the masses. The individual masses can be divided by the sum of the masses individually so that

$$center\ of\ mass = \sum_{n=1}^{N} x_n \cdot \frac{m_n}{M}. \tag{4-25}$$

If we define the normalized poundage at x_n as $P(x_n) = m_n/M$ (a number less than or equal to one), we can rewrite Equation 4-25 as

$$center\ of\ mass = \sum_{n=1}^{N} x_n\ P(x_n). \tag{4-26}$$

At the outset of this section we stated that expected value of a random signal is analogous to that of center of mass. Now that we have an understanding of center of mass we see that it is the value of x where we would expect to find all the mass of the system if they were all collocated and the balance of the system unchanged. To put this into the context of probability, let's again use Figure 4-10 but substitute probability for the normalized poundage, $P(x_n)$. Calling the center of mass for probability the expected value of X, Equation 4-26 becomes

$$E[X] = \sum_{n=1}^{N} x_n\ P\{x_n\}. \tag{4-27}$$

Now compare Equation 4-27 with Equation 4-19 where it is seen that the expected value of X is its statistical mean, i.e.,

$$E[X] = \overline{X} = \mu. \tag{4-28}$$

Note that two ways are used to denote the expected value of X: E[X] and $\overline{X}$.

For continuous random signals the expectation summation becomes integration so that

$$E[X] = \int_{-\infty}^{\infty} x\ f(x)\ dx. \tag{4-29}$$

It is instructive at this point to relate the expected or mean value of X to that of the time average of a time-varying signal. For comparison let

$$x(t) = A + \sin\left(\frac{2\pi}{T_0}\,t\right) \qquad 0 \le t \le T_0, \tag{4-30}$$

shown in Figure 4-11. This signal is seen to be a sine wave, one period in duration,

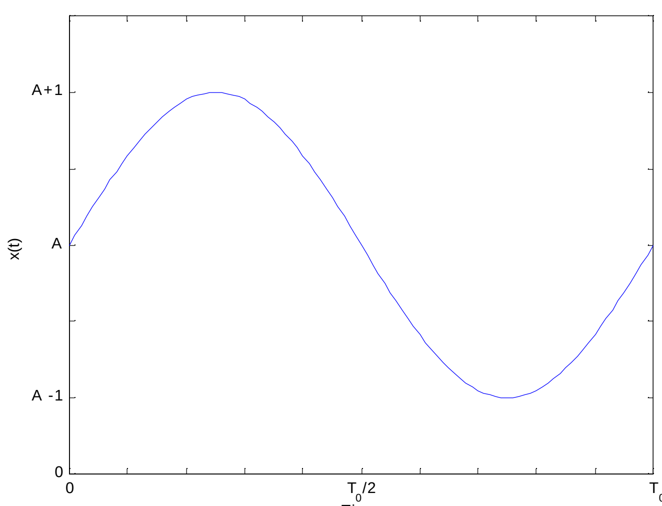


Figure 4-11. Sine wave + DC over one period.

riding on a DC value of A. The time average of x(t) is found using Equation 2-8 as

$$< x(t) > \ = \frac{1}{2T}\int_{-T}^{T} x(t)dt \ = \frac{1}{T_0}\int_{0}^{T_0}\left(A + \sin\left(\frac{2\pi}{T_0}\,t\right)\right)dt \ = A. \tag{4-31}$$

The time average of a sinusoid is always its DC value.

Now let's use Equation 4-29 to find the expected value of this waveform. Since the integration is over dx, the range of integration will be from the minimum value of x to its maximum. This range is seen to be from A – 1 to A + 1. The probability density of this waveform can be shown to be (see Gupta, for example)

$$f(x) \ = \frac{1}{\pi\sqrt{A^2 - x^2}}. \tag{4-32}$$

The expected value of this sinusoid is then

$$E[X] \ = \int_{-\infty}^{\infty} x\ f(x)\ dx \ = \int_{A-1}^{A+1} \frac{x}{\pi\sqrt{A^2 - x^2}}\ dx \ = A, \tag{4-33}$$

the same value as the time average found in Equation 4-31.

From this example we discern that the expected value of a time-varying signal, known as the statistical mean, produces the same result as the time average, which we know as the DC value. In other words, μ = DC for a time-varying signal. However, this is strictly true only for a class of signals called "ergodic". Unless otherwise stated we will assume all signals to be ergodic. Now we will see how extending the concept of expected value produces another important result.

We saw in Equation 4-29 that the expected value of X is found by multiplying the values of X (that is, the x values), by the probability of each of these values of x and summing the results. Expectation is not limited to just the center of mass of X. For example, we can find the expected value of X^2. This is found by squaring the values of x, multiplying the results by the probabilities of the individual values of x, and summing the outcome, i.e.,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \ f(x) \ dx. \qquad (4\text{-}34)$$

To see the physical significance of this expectation, again take x(t) as defined in Equation 4-30 and pictured in Figure 4-11. The average power of x(t) is found from Equation 2-9 as

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt = \frac{1}{T_0} \int_0^{T_0} \left[A + \sin\left(\frac{2\pi}{T_0} t\right)\right]^2 dt = A^2 + \frac{1}{2} \ W. \qquad (4\text{-}35)$$

Now, application of Equation 4-34 to find the expected value of X^2 of the signal of Equation 4-30 gives

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \ f(x) \ dx = \int_{A-1}^{A+1} \frac{x^2}{\pi\sqrt{A^2 - x^2}} \ dx = A^2 + \frac{1}{2}, \qquad (4\text{-}36)$$

the same result as from the time average of x^2 in Equation 4-35.

This example shows that the expected value of X^2 has the physical significance of being the average power of an ergodic time varying signal, i.e.,

$$P = E[X^2]. \qquad (4\text{-}37)$$

The last point to make in this section is that to generalize the mathematics of the expectation operator, the expected value of a function involving X, e.g., g(X), can be found as

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \ f(x) \ dx. \qquad (4\text{-}38)$$

4.3.2.2 Variance and Standard Deviation

With expectation we learned in the last section that the central weighted value of a random signal can be determined. This value is known as the statistical mean or average of the signal. While this is a useful and necessary component to be known of a random signal, it is not sufficient to adequately characterize it. In particular it does not quantify signal variability. In this section we will extend the use of $E[X^2]$ to quantitatively define signal scatter or variation.

Suppose three pieces of paper with the numbers 9, 10, and 11 are placed into a hat and drawn at random. The drawing from the hat produces random numbers and we can call the random number set X_1 so that $X_1 = \{9, 10, 11\}$. Since each number is equally likely, the probability of each is 1/3. From Equation 4-27 the expected value of X_1 is

$$E[X_1] = 9(1/3) + 10(1/3) + 11(1/3) = 10. \tag{4-39}$$

Now suppose another three pieces of paper with numbers 5, 10, and 15 are placed into another hat and drawn at random. Calling this set $X_2 = \{5, 10, 15\}$ we find

$$E[X_2] = 5(1/3) + 10(1/3) + 15(1/3) = 10, \tag{4-40}$$

the same as the first set. We see that even though the sets have the same expected value, they are not equivalent. In particular, they differ in their variability, i.e., the second set is scattered over a greater range.

One way to quantify variability is to square the random signal and find the expected value of the result, i.e., find $E[g(X)]$ where $g(X) = X^2$. For the first set we have

$$E\left[X_1^2\right] = \sum_{n=1}^{3} {x_{1_n}}^2 \, P\{x_{1_n}\} = 9^2(1/3) + 10^2(1/3) + 11^2(1/3) = 100.67. \tag{4-41}$$

Similarly for the second set

$$E\left[X_2^2\right] = \sum_{n=1}^{3} {x_{2_n}}^2 \, P\{x_{2_n}\} = 5^2(1/3) + 10^2(1/3) + 15^2(1/3) = 350. \tag{4-42}$$

So, it is seen that the greater variability of the second signal is confirmed by the expected value of the squares.

Now let a third hat contain paper pieces with the numbers 109, 110, and 111. It is seen that although the expected value (110) differs from the first set, its range of values is identical, that is, the value spread is over three consecutive numbers for both sets. Ideally, a measure of variability would return the same value for this set as for the first set. However, we find

$$E\left[X_3^2\right] = \sum_{n=1}^{3} x_{3_n}{}^2 \, P\{x_{3_n}\} = 109^2(1/3) + 110^2(1/3) + 111^2(1/3) = 12{,}101 \quad \text{(4-43)}$$

which is nowhere near the same as the expectation of the square of the first set.

The problem is that since the means are not the same squaring the signals squares the means as well. The solution is to subtract off the mean prior to squaring and then find the expectation. For the first set this results in

$$\begin{aligned} E\left[(X_1 - \mu_1)^2\right] &= \sum_{n=1}^{3} (x_{1_n} - \mu_1)^2 \, P\{x_{1_n}\} \\ &= (9-10)^2(1/3) + (10-10)^2(1/3) + (11-10)^2(1/3) = 2/3, \end{aligned} \quad \text{(4-44)}$$

a much smaller variability measure than that produced by Equation 4-41. Now subtracting the mean from the third set, squaring, and finding expectation yields

$$\begin{aligned} E\left[(X_3 - \mu_3)^2\right] &= \sum_{n=1}^{3} (x_{3_n} - \mu_3)^2 \, P\{x_{3_n}\} \\ &= (109-110)^2(1/3) + (110-110)^2(1/3) + (111-110)^2(1/3) = 2/3, \end{aligned} \quad \text{(4-45)}$$

exactly the same as set 1.

This exercise has shown that a consistent, useful measure of variability can be found by subtracting the mean from a random signal, squaring it, and finding the expectation of the result. This operation proves so useful and valuable it is given its own name and symbol, namely

$$variance = \sigma^2 = E[(X - \mu)^2] = \sum_{n=1}^{N} (x_n - \mu)^2 \, P\{x_n\}. \tag{4-46}$$

Since $(X - \mu)^2 = X^2 - 2\mu X + \mu^2$ and $E[aX + b] = aE[X] + b$, where a and b are constants, we find also that

$$\begin{aligned} \sigma^2 &= E[(X^2 - 2\mu X + \mu^2)] = E[X^2] - 2\mu E[X] + E[\mu^2] \\ &= E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2. \end{aligned} \tag{4-47}$$

This result is obtained by observing that $E[X] = \mu$ and $E[\mu^2] = \mu^2$. Notice the difference between Equation 4-37 and the outcome of Equation 4-47. In 4-37 we found that $E[X^2]$ is the total power, both AC and DC. However, for the variance, σ^2 is the total power minus the DC power, leaving only the AC power, i.e.,

$$\sigma^2 = P_{Total} - P_{DC} = P_{AC}. \tag{4-48}$$

Note that if $\mu = 0$, the DC power is zero and σ^2 represents the total power.

For continuous random signals, using Equation 4-38 the variance is found as

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 \, f(x) \, dx. \tag{4-49}$$

If the signal of interest is given in volts, then the units of variance are volts^2. It is often more convenient to work with the original units, that is volts, so the square root of the variance is commonly used instead, called the standard deviation. Therefore,

$$standard\ deviation = \sigma = \sqrt{\sigma^2} = \sqrt{E[(X - \mu)^2]}. \tag{4-50}$$

But, since σ^2 represents AC power, as given by Equation 4-48, σ gives the RMS voltage of the time-varying signal.

We now have developed two tools which allow us to describe random signals, namely the expected value and variance (or equivalently the standard deviation). However, other than a measure of variability, we have no way of describing how the voltage (or current) of the random signal is distributed. To see why this is necessary consider one last example. Take two 18-element random signals X_1 = {1, 7, 8, 8, 9, 9, 9, 10, 10, 10, 10, 11, 11, 11, 12, 12, 13, 19} and X_2 = {5, 6, 6, 6, 7, 7, 7, 9, 10, 10, 11, 13, 13, 13, 14, 14, 14, 15}, shown plotted in Figure 4-12 with X_1 in the upper plot and X_2 in the lower. Application of Equations 4-27 and 4-46 reveals that μ = 10 and σ^2 = 11.84 for both X_1 and X_2. From the plots it is obvious that the two signals are not identical, even though their statistical descriptors are. The obvious difference is that the two signals are distributed differently, as shown in Figure 4-13. With a plot of the distribution it is possible to determine the range of values, concentration of energy, etc. This concept was also presented when we introduced probability density functions in Section 4.3.1.2. We will now examine probability density functions analytically using the tools we developed in this section.

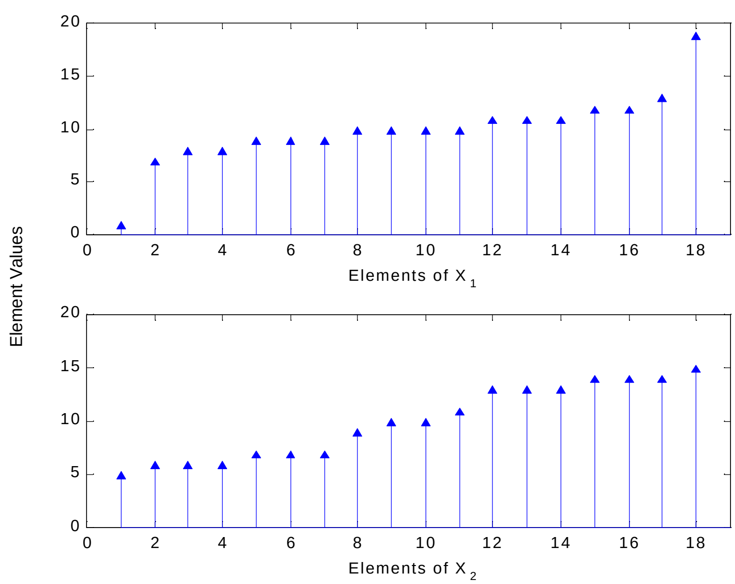


Figure 4-12. Values of the elements of X_1 on top and X_2 below.

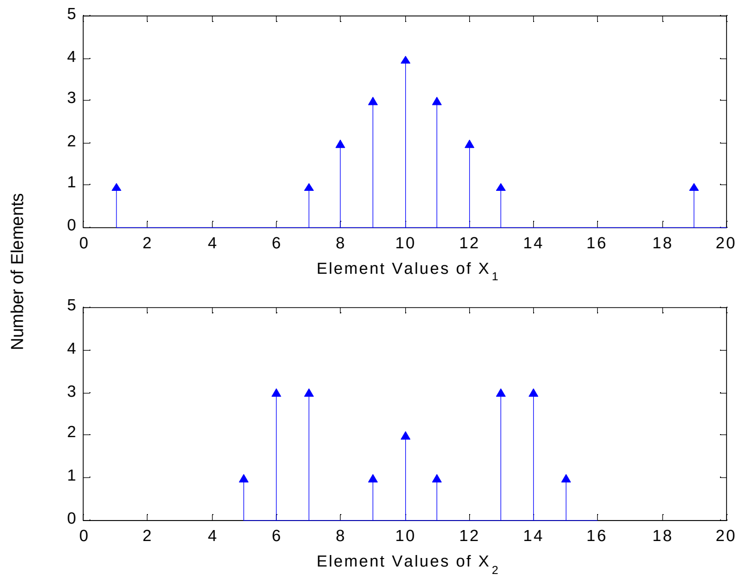


Figure 4-13. Distribution of element values of Figure 4-12.

4.4 SOME PROBABILITY DISTRIBUTIONS

The concept of the probability density function was introduced in Section 4.3.1.2. Along with μ and σ described in the last section, the pdf can be used to distinguish a random signal's statistical characteristics. Its primary utility is in demarcating the range of values (e.g., volts) over which a random signal can be found and the concentrations of energy within that range. By summing up the concentrations of energy over a range of interest, the probability of the signal's value falling within that range of interest can be determined.

As an example of using a pdf to find probability suppose we wish to determine if a gallon of gas pumped at a gas station is actually a gallon. And, if it is not, what are the range of values we might find and the probability of a given range of values. To set up the problem, let g represent the actual amount of gas pumped at a station with the gas pump reading exactly 1.0 gallon. Since we wish to determine how far this measurement differs from 1.0 gallon, let the difference be $x = |g - 1|$. We include the absolute signs since we are not concerned with whether the error reading is high or low, just by how much it differs from exactly one gallon.

Say it is known that the probability density function of how the measurement differs from the reading is known as

$$f(x) = \begin{cases} 10\ e^{-10x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{4-51}$$

This pdf is shown plotted in Figure 4-14, where it is seen that the concentration of measurements is greatest at x = 0 (indicating no difference between measurement and reading), and decreasing with greater values of x. The range of measurement differences is seen to vary from 0 to about 0.55 gallons.

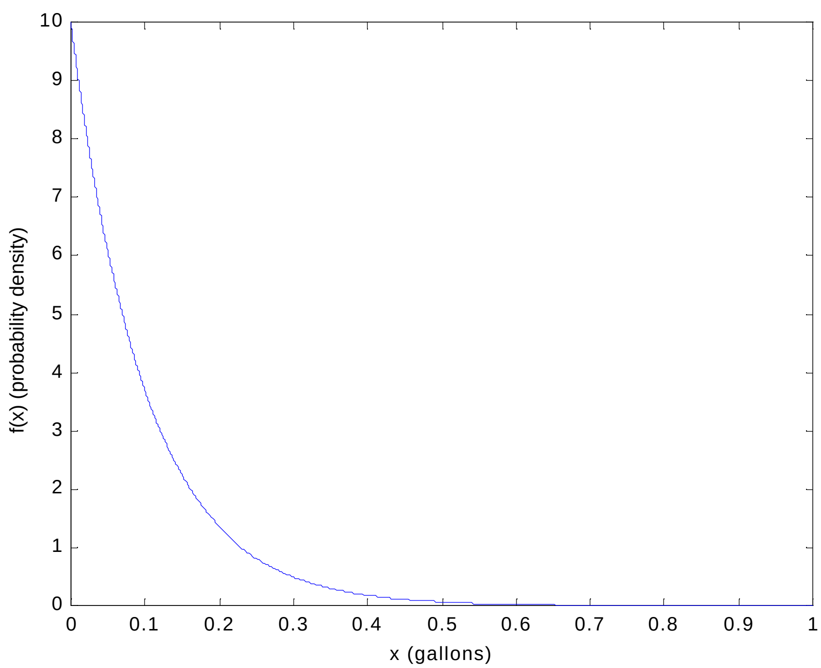


Figure 4-14. Probability density function of difference between actual and measured gallon of gas.

Since x is the actual measurement error values, we can refer to the random variable associated with the experiment as X. If we wish to know the probability of a measurement error being less than some value of x, i.e., $P\{X \leq x\}$, we must sum all the contributions from the density function at values less than or equal to the chosen value of x. Since this pdf is a continuous function, this summation is an integration. This probability is computed so frequently that it has its own name, the cumulative distribution function (cdf) with symbol F(x). In mathematical terms this is given as

$$F(x) = P\{X \leq x\} = \int_{-\infty}^{x} f(x)\ dx. \tag{4-52}$$

This equation can be used if, for example, we wished to know the probability that the measurement would differ by 0.3 gallon or less, we set $x = 0.3$ and find

$$P\{X \leq 0.3\} = \int_{0}^{0.3} 10\ e^{-10x}\ dx = 1 - e^{-10(0.3)} = 0.95. \tag{4-53}$$

The lower limit of integration is zero since the pdf is zero for $x < 0$. If we instead wanted the probability that the measurement error exceeded some value x, it is seen that

$$P\{X \geq x\} = \int_{x}^{\infty} f(x)\ dx. \tag{4-54}$$

We could use this equation to find the probability that the measurement is in error by more than 0.5 gallon as

$$P\{X \geq 0.5\} = \int_{0.5}^{\infty} 10\ e^{-10x}\ dx = e^{-10(0.5)} = 0.0067. \tag{4-55}$$

Finally, to find the probability that the error value falls between two values x_1 and x_2, where $x_1 < x_2$, it follows that

$$P\{x_1 \leq X \leq x_2\} = \int_{x_1}^{x_2} f(x)\ dx = F(x_2) - F(x_1). \tag{4-56}$$

The expected value of the error measurement is found using Equation 4-29 as

$$E[X] = \int_{-\infty}^{\infty} x\ f(x)dx = \int_{0}^{\infty} 10x\ e^{-10x}\ dx = -\frac{e^{-10x}}{10}\ (10x - 1)\Bigg]_{0}^{\infty} = \frac{1}{10}. \quad (4\text{-}57)$$

This result indicates that the expected error in measurement is 0.1 gallon for every gallon pumped. Using Equations 4-47 and 4-50, the standard deviation is also found to be 0.1 gallon.

Now that the relationship between probability, the cdf, and the pdf has been illustrated, we will now describe some probability density functions that we will find useful in studying noise effects on receiver systems.

4.4.1 Uniform Probability Density

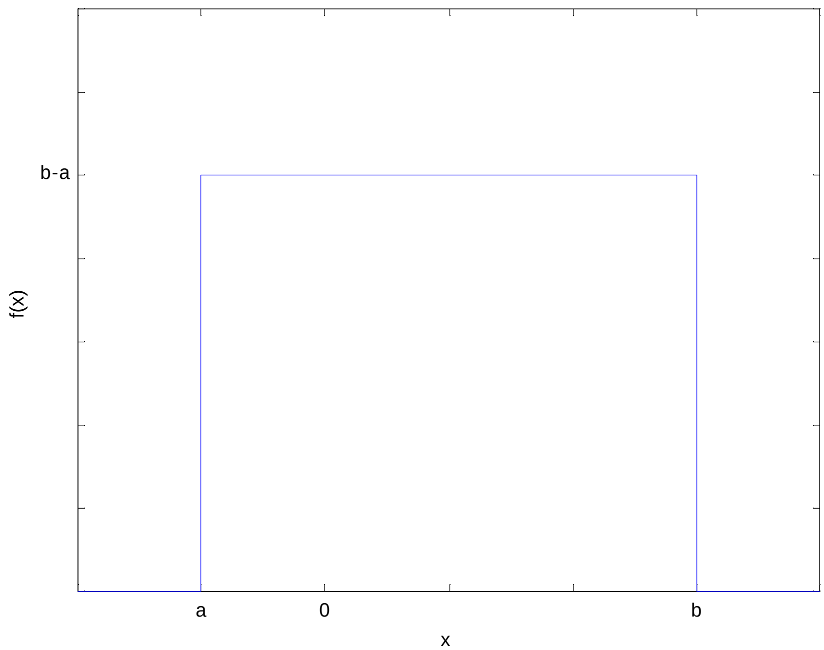
The uniform probability has a constant or flat value over its range. The probability of an interval within that range is proportional to the interval length. This implies that all selections from the range are equally probable. Think or a roulette wheel. After it is spun the pointer is just as likely to stop on one number as any other. The probability density function is given as

$$f(x) = \frac{1}{b - a}, \quad a \le x \le b \quad (4\text{-}58)$$

and zero otherwise, as shown plotted in Figure 4-15. Using Equation 4-52, the cdf is found as

$$F(x) = \frac{x - a}{b - a}, \quad a \le x \le b. \quad (4\text{-}59)$$

The mean and variance of the uniform density can by found by application of Equations 4-29 and 4-49 to find



(4-60)

Figure 4-15. Uniform probability density function.

The primary uses of the uniform distribution are first to determine the probability of a random phase of a signal, where it is usually assumed that a random phase has a uniform distribution over $-\pi/2$ to $\pi/2$ or from 0 to 2π, and second as a generating function for other pdfs in simulation.

4.4.2 Normal or Gaussian Probability Density

The most used and therefore most important pdf is the Gaussian, first advanced by Abraham Demoivre in 1733. Many naturally occurring phenomena, including thermal noise, are very accurately modeled with the Gaussian pdf. Its modeling equation with mean μ and variance σ^2 is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} . \tag{4-61}$$

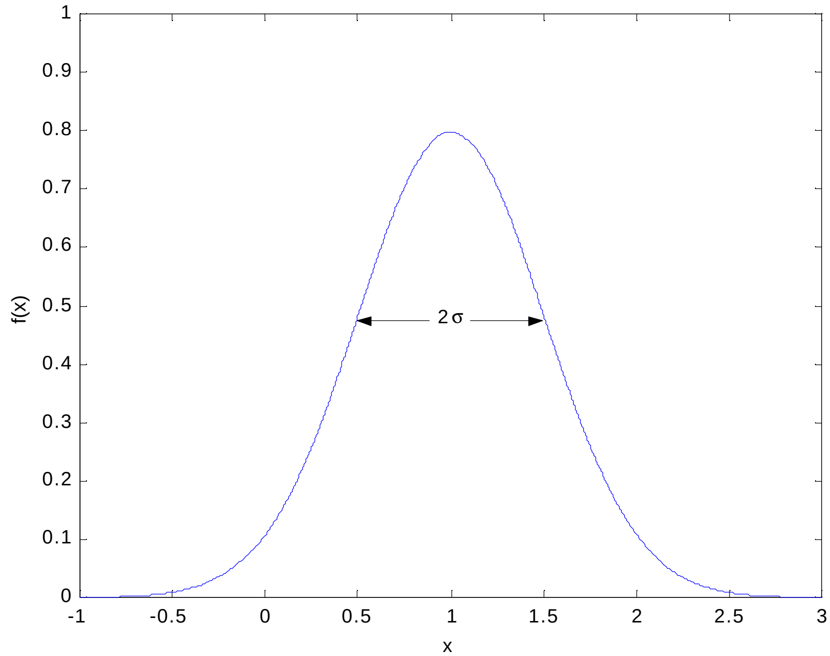


Figure 4-16. Gaussian pdf with $\mu = 1$ and $\sigma = 0.5$.

A plot of the gaussian pdf with $\mu = 1$ and $\sigma = 0.5$ is shown in Figure 4-16. Note that the Gaussian pdf represents the well-known "bell curve".

Recall from Equation 4-56 that to determine probability the pdf must be integrated to find the cdf. Unfortunately Equation 4-61 cannot be integrated in closed form to give an equation for F(x). Therefore, numerical integration must be used to find the required probability.

To preclude the need for numerical integration, tables can be created and consulted when necessary for probability computations. However, to allow for all conceivable values of μ and σ, the number of tables required would be prohibitive. Therefore, a "standard" Gaussian density is used where $\mu = 0$ and $\sigma = 1$. Integration of the standard Gaussian density as per Equation 4-61 is given the symbol $\Phi(x)$, and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} \, dx. \tag{4-62}$$

Tables of Φ(x) are found in many texts on probability or statistics. For arbitrary values of μ and σ, these tables can be used with a simple change of variables to find the desired cdf of

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right). \tag{4-63}$$

Some texts list tabulated values of Q rather than Φ where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{x^2}{2}} \, dx. \tag{4-64}$$

Since the total probability is equal to 1 it is seen that

$$Q(x) = 1 - \Phi(x). \tag{4-65}$$

Other references (particularly mathematics books) tabulate values of the error function, defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-x^2} \, dx. \tag{4-66}$$

The complementary error function is the integral from x to ∞ so is related to the error function as erfc(x) = 1 – erf(x). The error function can be associated with the Q function using

$$Q(x) = \frac{1}{2}\left[1 - erf\left(\frac{x}{\sqrt{2}}\right)\right]. \tag{4-67}$$

4.4.3 Log-Normal Probability Density

The log-normal distribution of x is similar to the normal distribution of x with the difference being that here the logarithm of x is normally distributed. We will use the natural logarithm of x, ln(x), although logarithms of any base could be used.

If we set Y = ln X, then it is readily seen that Y is a Gaussian random variable with mean μ_Y and variance σ_Y^2. A pdf transformation from Y to X results in

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_Y x} e^{-\frac{(\ln(x) - \mu_Y)^2}{2\sigma_Y^2}} \qquad x \geq 0. \tag{4-68}$$

A plot of the log-normal pdf is shown in Figure 4-17 with $\mu_Y = 1$ and $\sigma_Y^2 = 0.5$.

The cdf cannot be found in closed form, but standard Gaussian tables can be used with the substitution of ln(x) for x,

$$F(x) = \Phi\left(\frac{\ln(x) - \mu}{\sigma}\right). \tag{4-69}$$

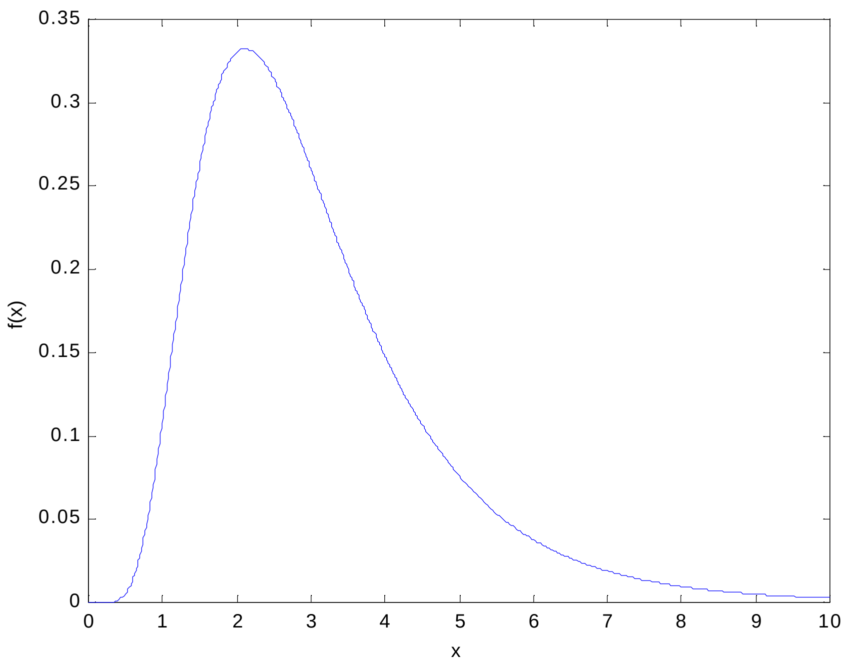


Figure 4-17. Log-normal pdf with $\mu_Y = 1$ and $\sigma_Y^2 = 0.5$.

The mean and variance are found (after some careful calculations) as

$$\mu_X = e^{\mu_Y + \frac{\sigma_Y^2}{2}} \quad and \quad \sigma_X^2 = e^{2\mu_Y}\left(e^{2\sigma_Y^2} - e^{\sigma_Y^2}\right). \tag{4-70}$$

4.4.4 Gamma Probability Density

The gamma density is a versatile pdf which finds widespread use in predicting message length, message arrival times, reliability, and failure rates. Its definition includes the gamma function, from which it gets its name. The pdf, with parameters $\alpha > 0$ and $\beta > 0$, is

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \qquad x > 0, \tag{4-71}$$

where the gamma function $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy = (\alpha - 1)\Gamma(\alpha - 1). \tag{4-72}$$

A plot of the gamma pdf is shown in Figure 4-18 with several values of α and β.

As seen in the plot, the pdf shapes can change so much as a function of the parameters that they appear to have been created by different functions. Because of the unique attributes of individual pdf shapes, several have assumed names of their own. For example, for $\alpha = 1$ the gamma pdf is known as an exponential, used widely for time-of-arrival studies. Extending α to include all (positive) integers the gamma is known as an erlang, which finds use by actuaries and in predicting mean time between failures. One other common pdf, often used in meteorology, is chi-squared. It is formed by setting $\alpha = n/2$ and $\beta = 2$ where n is a positive integer.

In general, (except for the exponential pdf) the cdf cannot be determined in closed form; numerical integration must be used. The gamma mean and variance are

$$\mu = \alpha\beta \qquad and \quad \sigma^2 = \alpha\beta^2. \tag{4-73}$$

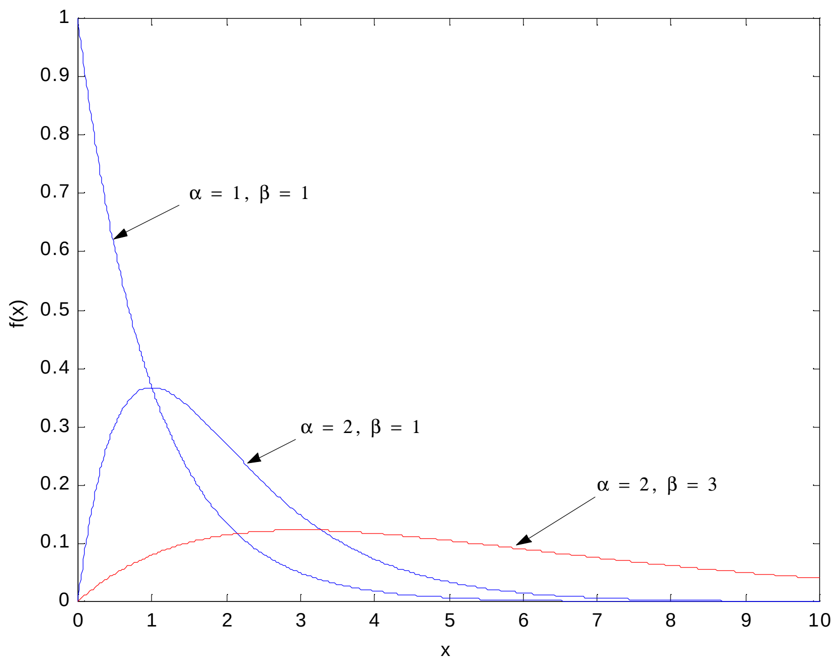


Figure 4-18. Gamma pdf with three different parameter sets.

4.4.5 Weibull Probability Density

The final pdf we will examine is the Weibull, sometimes called a "power Rayleigh". Like the gamma, it is also used for failure rate prediction, but it is considered easier to use since the gamma function is not included and it can be integrated in closed form. Its pdf with parameters α and β is given by

$$f(x) = \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-(x/\beta)^{\alpha}} \qquad x > 0. \tag{4-74}$$

The cdf is easily found to be

$$F(x) = 1 - e^{-(x/\beta)^{\alpha}} \qquad x > 0. \tag{4-75}$$

Its mean and variance are related to the gamma function, however. They are given by

$$\mu = \beta\Gamma\left(1 + \frac{1}{\alpha}\right) \quad \textit{and} \quad \sigma^2 = \beta^2\left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right)\right]. \tag{4-76}$$

The Weibull pdf is shown plotted in Figure 4-19 for several parameter values. Notice the shape change for the different parameters. Just as with the gamma, when $\alpha = 1$ the function reduces to the exponential distribution. For the case when $\alpha = 2$, the Weibull is called a Rayleigh. The Rayleigh is commonly used for miss distance prediction of bombs, bullets, arrows, darts, etc., and narrow-band noise envelope.

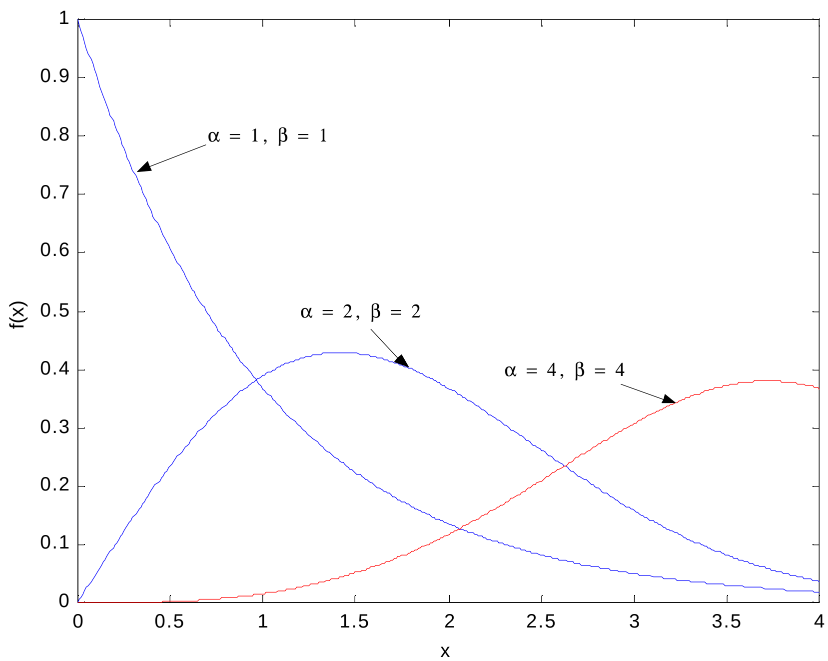


Figure 4-19. Weibull pdf with three different parameter sets.

4.5 SIMULATED NOISE GENERATION

In the last section we examined several probability density functions which are mathematical models of random events occurring in nature. Our purpose in this section is to simulate noise signals whose voltage amplitudes are distributed according to a chosen pdf. We will use the noise signals developed in this section to represent jamming signals used against radio receiving systems in Chapters xxx.

4.5.1 Noise Generated from Probability Density Function

The probability distributions discussed in Section 4.4 can be used to generate noise signals whose distributions are representative of the pdfs that created them. In other words, a signal created from a normal pdf, for example, will have a normal distribution while one created from a gamma pdf will have a gamma distribution.

The procedure to find the values of a noise signal is the inverse of finding the probability of x given x, that is to find F(x) knowing x. In this inverse case we assume we know F(x) and from that we will determine x. Mathematically, it is seen that $x = F^{-1}[F(x)]$. As an example suppose $F(x) = x^2$, then $F^{-1}(y) = y^{1/2}$. If follows that $F^{-1}[F(x)] = F^{-1}[x^2] = [x^2]^{1/2} = x$. The problem becomes one of finding F^{-1}, the inverse of F. In some cases it is analytic and simple, in others we will employ the computational power of the computer to help determine the inverse of F.

If we are given a value of F(x) and the inverse of F, we can determine the x value from which the F(x) value came. In other words

$$if \quad x \xrightarrow{F} F(x), \qquad then \qquad F(x) \xrightarrow{F^{-1}} x. \tag{4-77}$$

Now, we assume that all values of F(x) are equally likely, that is the values of F(x) are themselves uniformly distributed. Then if we generate a uniformly distributed random signal (available in many computer applications) and let those values represent the values of F(x), application of F^{-1} to those values will result in a random signal which is randomly distributed the same as F(x).

As an illustration suppose f is Weibull distributed with $\alpha = \beta = 2$ so that $F = 1 - e^{-(x/2)^2}$. Solving for x we find $F^{-1} = 2\,[-\ln(1 - F)]^{1/2}$. If we now generate a uniformly distributed signal with elements of say, 0.6154, 0.7919, 0.9218, 0.7382, and 0.1763, the corresponding Weibull random signal will have elements of 1.9551, 2.5059, 3.1929, 2.3153,and 0.8807.

Figure 4-20 shows examples of simulated noise generated from normal, log-normal, gamma, and Weibull probability distributions. These pdfs are shown in the four left hand plots. All the distributions shown have the same mean and variance of 2.5 and 4 respectively. The right hand plots show the time domain signals created from their adjacent pdfs. These signals appear similar because they were all created using the same uniform random signal which was transformed by $F^{-1}(x)$ for each distribution. There are observable differences among them however. For example, the signal generated from the normal pdf spends most of the time near and is balanced around the mean while the others are concentrated at different locations and are not balanced.

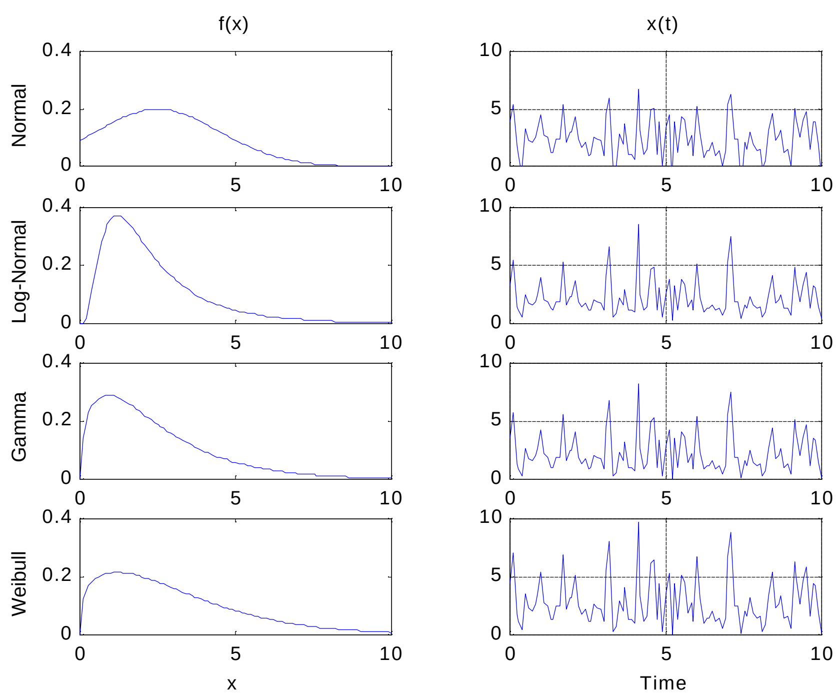


Figure 4-20. Four probability density functions and their associated noise signals.

4.5.2 Simulated Flicker Noise

In Section 4.2.3 we introduced flicker noise and discussed that its PSD diminished as 1/f. We can simulate flicker noise by starting with any white noise source (e.g., thermal noise) and passing it through a filter which has a frequency response of 1/f.

As an example, suppose we have the white noise shown in the top plot of Figure 4-20 above. If we pass this noise signal through a filter with a frequency response of 1/f we get the new noise signal shown in Figure 4-21, where the original signal is shown on top and the filtered signal on bottom. Notice that the high frequency components of the original signal have been removed by filtering. In Section 4.6 we will confirm this observation in the frequency domain.

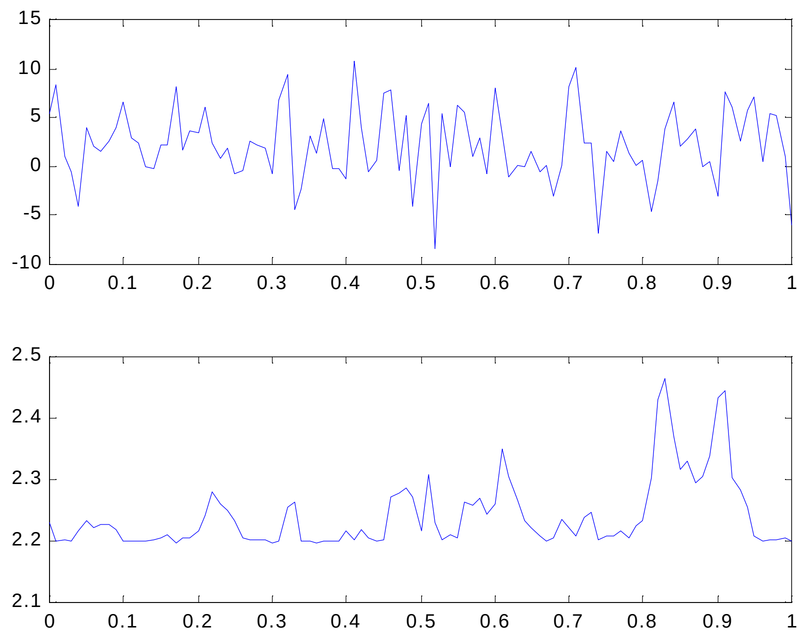


Figure 4-21. Gaussian noise, shown on top, and after passing through a 1/f filter, shown on the bottom.

4.5.3 Simulated Impulsive Noise

The final noise we wish to simulate is impulsive noise. Atmospheric noise and a number of man-made noises are impulsive. Being impulsive this type noise arrives in bursts or pulses rather than continuously as depicted in the previous examples. For this reason it cannot be modeled with the continuous probability density functions. Instead, we must use other models.

First, we observe that the time between bursts or pulses, called interarrival time, is random. Interarrival times are best predicted by gamma or weibull distributions. Second, the magnitude and phase of the arriving burst is also random, with any of the probability distributions probably good models. Finally, the pulse can take any shape we wish to ascribe to it. Jeruchim, et al. suggest using a gamma distribution for interarrival times, a log-normal or a Weibull distribution for pulse amplitude, a uniform distribution for pulse phase, and a decaying exponential with or without accompanying sinusoid for pulse shape. Figure 4-22 depicts simulated impulsive noise with gamma interarrival times and Weibull amplitudes. The top plot shows an impulsive pulse shape, the middle an exponential impulse, and the bottom represents an exponential impulse with accompanying sinusoid.

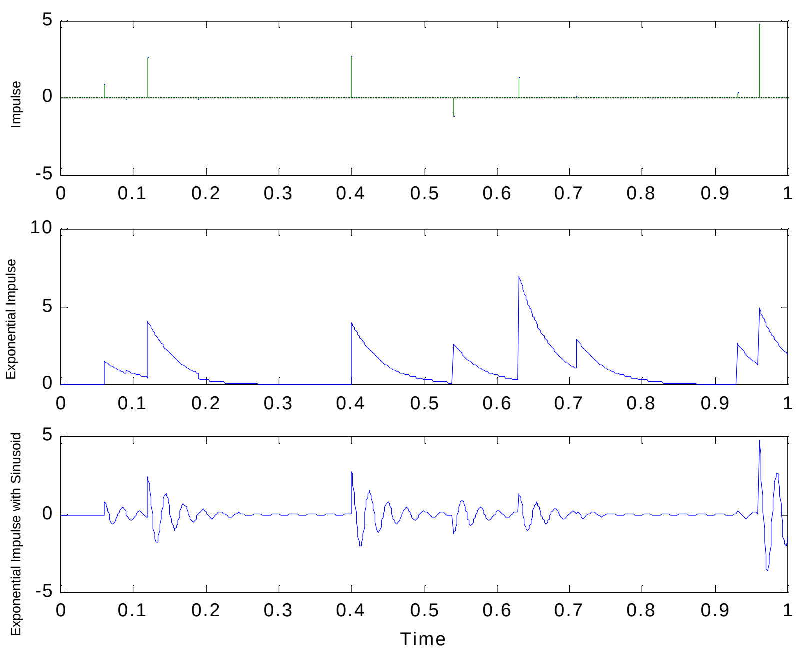


Figure 4-22. Simulated impulsive noise with interarrival times gamma and amplitudes Weibull distributed. The top plot represents a purely impulsive pulse shape, the middle an exponential impulse, and the bottom an exponential impulse with accompanying sinusoid.

As noted in Sections 4.1 and 4.2, radio receivers at VHF and above are generally most susceptible to thermal (gaussian) noise. This means that most receivers are designed to operate optimally in the presence of thermal noise, disregarding other noise types. Impulsive noise can be a formidable jamming waveform against such receivers as we will see in Chapter xxx.

4.6 NOISE POWER SPECTRAL DENSITY

In Section 3.6 we found the power spectral density (i.e., the distribution of power in frequency) of deterministic signals and in Section 4.2.1 we determined the PSD for thermal white noise. Here we will discern the general-case PSD for noise or random signals, irrespective of their sources or pdfs.

For a time-domain signal, whether deterministic or random, the total energy in the signal is stated in Equation 2-7, repeated here as

$$E = \int_{-\infty}^{\infty} x^2(t)\ dt. \tag{4-78}$$

In Section C.5 it is shown that the total energy for the same signal, but represented in the frequency domain, is

$$E = \int_{-\infty}^{\infty} x^2(t)\ dt = \int_{-\infty}^{\infty} |X(f)|^2\ df, \tag{4-79}$$

where X(f) is the Fourier transform of x(t) and $|X(f)|^2$ is the square of the magnitude of X(f).

The average power of x(t) is found using Equation 2-9, and combined with Equation 4-79, we have

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |X(f)|^2 \; df$$
$$= \int_{-\infty}^{\infty} \left[\lim_{T \to \infty} \frac{1}{2T} \; |X(f)|^2 \right] df. \tag{4-80}$$

Comparing Equations 4-80 and 3-27 illustrates that the average power is

$$P = \int_{-\infty}^{\infty} \left[\lim_{T \to \infty} \frac{1}{2T} \; |X(f)|^2 \right] df = \int_{-\infty}^{\infty} S_x(f) \; df, \tag{4-81}$$

so that the power spectral density of x(t) is given by

$$S_x(f) = \lim_{T \to \infty} \frac{1}{2T} \; |X(f)|^2. \tag{4-82}$$

The results obtained here evidence that for an arbitrary signal x(t) within a time interval $-T \le t \le T$, the power spectral density can be approximated by calculating the Fourier transform of x(t), squaring the magnitude of the result, and dividing by the length of the interval. As T approaches infinity, the results become exact rather than an approximation.

As an illustration of varying the length of the interval, we can compute the PSD of a thermally generated signal. We know that a thermal source produces white noise so we expect its PSD to be flat or constant across the frequency spectrum. Figure 4-23 shows the results of these computations where we vary the length of the interval I over which the computations are based. In the top plot the interval is 2^{10}, in the center it is 2^{15}, and in the bottom plot it is 2^{20}. It is easily seen that as the interval increases in length, the more the PSD becomes flat. In the limit, that is when the interval length is infinity, the PSD for white noise becomes perfectly flat.

In Section 4.5.2 we discussed 1/f or pink noise and observed a time domain representation of a pink noise signal which had been created by filtering a white

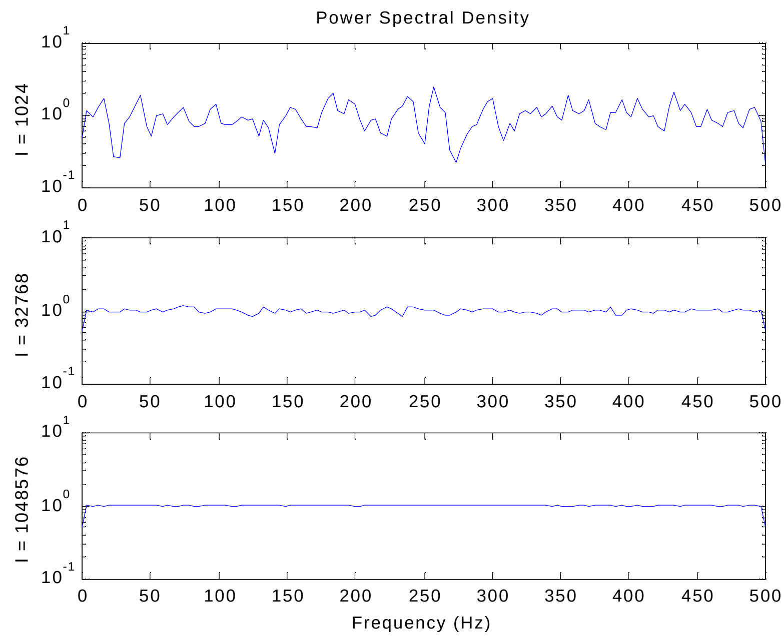


Figure 4-23. White noise PSDs for different interval lengths. The length in the top plot is 2^{10}, middle 2^{15}, and the bottom 2^{20}.

noise signal. The PSD of this pink noise signal is shown in Figure 4-24 where we see how the power distribution differs from the white noise PSD.

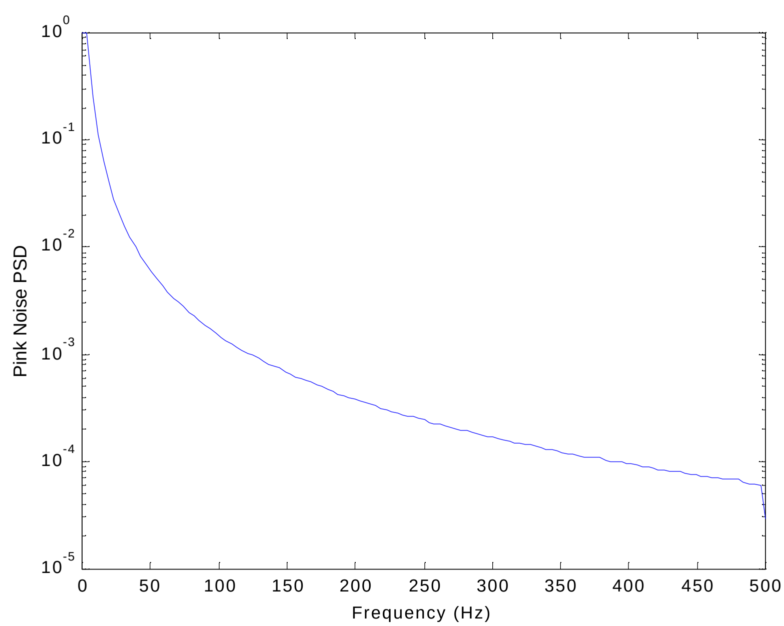


Figure 4-24. Pink noise PSD.

Knowing how to obtain the PSD gives us the ability to find the average power of a noise signal, by using Equation 4-81. Assuming we already know the power contained in a communications signal, we can find the ratio of the signal power to noise power, known as the signal-to-noise ratio.

4.7. SIGNAL-TO-NOISE RATIO

Whether we are concerned with analog or digital signals, the fundamental predictor of signal quality at the receiver is the signal-to-noise ratio. Therefore, in signal reception analysis, it is mandatory to determine the signal-to-noise ratio. (In Chapter xxx we will modify this to signal-to-jammer ratio.) The signal-to-noise ratio (SNR) is the ratio of the signal power to the noise power and is a dimensionless number. It is sometimes indicated in this form, but the most common expression of the SNR is in decibels. This is found in the usual way,

$$SNR_{dB} = 10\ \log(SNR). \tag{4-83}$$

There are many locations within a receiver where the SNR can be found, and although the values found are related they are not identical. For example, the SNR at the input of the receiver will be greater than at the input of the demodulator because of added noise such as flicker and shot noise within the receiver. This will be discussed more fully in Chapter xxx where the relationship between received signal quality and SNR will be elaborated.